Combinatorial enumerations and Graycodeness on restricted growth functions avoiding vincular patterns

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Abstract—The manuscript presents enumerations for restricted growth functions (RGFs) avoiding (i) single vincular pattern sets of length at most 3 and (ii) two-pattern sets of a vincular pattern and a classic pattern of length at most 4. The presented enumerations are counted by known classic sequences like Bell, Fibonacci, binary strings, etc. Moreover, we show a sufficient condition for the 3-Graycodeness by the reflected Gray code order of RGFs avoiding a pattern set. Consequently, it allows us to prove the 3-Graycodeness for some classes of RGFs avoiding particular pattern sets in (i) and (ii) by checking the condition satisfaction. This, in many cases, helps to reduce remarkably routine work for proving the 3-Graycodeness on RGFs.

I. INTRODUCTION

The field of pattern avoiding permutations has been showing an increasing interest in the last two decades. Many variants of classic pattern avoidance have been proposed for enumerations on pattern avoiding objects. Vincular pattern avoidance is one of them. It was introduced by Baxter and Pudwell in [4] on permutation sets and words. Vincular patterns resemble classic patterns with the constraint that some of the letters in a copy must be consecutive. An occurrence of a vincular pattern in a word requires the adjacency in the position of corresponding letters as consecutive letters in the pattern. By that Baxter and Pudwell gave general enumeration schemes which allow to show many Wilf-Stanley equivalence classes on permutations avoiding different vincular pattern sets.

Although vincular patterns have been investigated intensively on permutation sets, there are not so many results given on restricted growth functions (RGFs). The purpose of this manuscript is precisely to study RGFs avoiding vincular patterns. In particular, given a vincular pattern set $T$ we are interested in the two following problems on the set of RGFs of length $n$ avoiding all patterns in $T$, denoted by $R_n(T)$:

(i) enumerating numbers of elements of $R_n(T)$ for $n = 1, 2, \ldots$.

(ii) investigating the Graycodeness on $R_n(T)$.

In [12], Sagan showed that the set of RGFs of length $n$ is naturally bijective to set partitions of $\{1, 2, \ldots, n\}$. Since then studies about combinatorial enumerations, distributions of statistics, and generating algorithms on RGFs avoiding classic pattern sets have blossomed [5]–[8], [12]. For our problem (i), there may have numerous possible cases for considering a vincular pattern set $T$. In this paper, we consider only vincular patterns of length $\leq 4$ with two adjacent letters. Addition to that pattern sets $T$ considered are ones of a single vincular and ones of a vincular pattern and a classic pattern. Our enumeration results on $R_n(T)$ are counted by known classic sequences like Bell, Fibonacci, odd indexed Fibonacci sequences, etc.

For our problem (ii), the Graycodeness (resp. $d$-Graycodeness) allows to order words of a given class linearly such that Hamming distances between any two consecutive elements are at most 1 (resp. $d$). Studies the Graycodeness on combinatorial objects have received much attention because of its numerous applications in error-detecting codes and efficient generating algorithms designs as well. For sets of RGFs with no pattern avoidance and bounded RGFs of length $n$, Sabri and Vajnovszki in [10], [11] showed their 3-Graycodenes using reflected/co-reflected Gray code orders. In this paper, we show a simple sufficient condition for being 3-Graycode by the reflected order of $R_n(T)$ basing on generation rules. Consequently, we can prove the 3-Graycodeness on some classes of RGFs avoiding vincular patterns by checking the condition satisfaction. This helps to reduce remarkably routine work for proving the 3-Graycodeness on RGFs.

II. ENUMERATIONS ON VINCULAR PATTERN AVOIDING RGFs

In this section, we present enumerations for restricted growth functions (RGFs) avoiding a vincular pattern and avoiding a set of a vincular pattern and a classic pattern. These enumerations show their relations to known classical sequences in Sloane [13] like Bell, Fibonacci, binary strings, etc.

Denote $[n] = \{1, 2, \ldots, n\}$ the set of consecutive numbers from 1 to $n$. We first recall some basic concepts on RGFs.

Definition 1. Let $s = s_1s_2\ldots s_n$ be a word of non-negative integers of length $n$. Then $s$ is called a restricted growth
function of length \( n \) (or an RGF) if \( s_1 = 0 \) and \( s_i \leq \max \{s_1, s_2, \ldots, s_{i-1} \} + 1 \) for any \( i = 2, \ldots, n \).

Notice that in the literature, by definition, restricted growth functions often start with 1. However, for the sake of simplicity when considering their Grayorder in Section III, restricted growth functions in our definition always start with 0.

A (classic) pattern is a representation of a word using the relative ordering, where its ith smallest letter is represented by \( i \). For example, the word 34727 has the pattern 23414. For a given pattern \( \tau = \tau_1 \tau_2 \ldots \tau_n \) and a word \( s = s_1 s_2 \ldots s_n \) where \( n \geq k \), an occurrence of the pattern \( \tau \) in \( s \) is a \( k \)-subword \( s_i s_{i+1} \ldots s_{i+k-1} \) for \( 1 \leq i < i_2 < \cdots < i_k \leq n \), of \( s \) that is order-isomorphic to \( \tau \). If \( s \) does not contain any occurrences of \( \tau \), then we say that \( s \) avoids \( \tau \). For instance, \( 123 \) occurs 4 times in the word \( 0121203 \).

Vincular patterns resemble classic patterns in which some adjacent letters are underlined. An occurrence of a vincular pattern \( \tau \) with the set of underlined letters \( X \) in a word is an occurrence of \( \tau \) which requires the adjacency in the position of letters in the subword corresponding to underlined letters \( X \) in \( \tau \). For example, in the word \( 0121203 \), the pattern \( 123 \) occurs as subword 223 (since 22 are adjacent), but not as 003 and 113. In general, if a pattern \( \tau \) contains \( \tau_1 \tau_{1+1} \ldots \tau_j \), then the letters corresponding to \( \tau_1, \tau_{1+1}, \ldots, \tau_j \) in a word must be adjacent.

Given a pattern set \( T \) (classic or vincular does not matter), we denote by \( R_n(T) \) the set of all RGFs of length \( n \) avoiding all patterns in \( T \). By default, \( R_n \) is the set of all RGFs of length \( n \) where \( T = \emptyset \). Let \( R_n \) be the cardinalities of \( R_n \), \( n \geq 0 \) are counted by Bell numbers (the sequence \( A000110 \) in Sloane [13]). Enumerations of RGFs avoiding a classic pattern or length 3 are given in [12].

The following shows characterizations and enumerations for RGFs avoiding a vincular pattern of length 2 or 3.

**Theorem 1.** Let \( n \) be a positive integer. Then,

1. \( R_n([11]) \) is the set of RGFs of length \( n \) with no two adjacent equal letters. Additionally, \( |R_n([11])| \) is counted by the \((n-1)\)th Bell numbers.
2. Let \( \tau \) be one of the following vincular patterns \( \{21, 121, 12, 122, 123, 22\} \). Let \( \tau' \) be the classic pattern obtained from \( \tau \) by removing the adjacency in \( \tau \). Then, \( R_n(\tau') = R_n(\tau') \). Consequently, \( |R_n(\tau)| = 2^{n-1} \).
3. \( R_n([12]) \) is the set of RGFs which do not contain any two adjacent letters for any \( n \geq 1 \). Furthermore, \( |R_n([12])| \) is counted by the sequence \( A261041 \) in Sloane.

**Proof.**

1. The first statement is trivial from the definition of vincular pattern avoidance. For the second statement, we recall in [6] that each \( s \in R_n \) one-to-one corresponds to a set partition \( \{B_1, B_2, \ldots, B_k\} \) of \( [n] \) where \( B_k = \{i: s_i = k - 1\} \). For instance, the corresponding set partition for \( 00101223 \) is \( \{\{1, 2, 4\}, \{3, 5\}, \{6, 8\}, \{7\}\} \). So if \( s \in R_n([11]) \), then all blocks of the corresponding set partition of \( s \) do not contain any two consecutive numbers. By [9], the number of partitions of \( [n] \) satisfying this condition counts the \((n-1)\)th Bell number.
2. The statement is followed from the definition of vincular pattern avoidance on RGFs. More precisely, it is easy to characterize RGFs avoiding a vincular pattern \( \tau \) given in the statement. Hence, enumeration results are then implied as consequences of results for enumerating RGFs avoiding corresponding classical patterns of length 3 given by Sagan in [12].
3. The first statement is trivial from the definition of the vincular pattern. For the second one, we will use the correspondence between RGFs and set partitions of \( n \) as in the proof of part (i). If \( s \in R_n([12]) \), then by definition its corresponding blocks \( \{B_2, B_3, \ldots, B_k\} \), except possibly the block \( B_1 \) (where it is possible to accept two adjacent 0s in \( s \), in the set partition of \( s \) do not contain two consecutive numbers.

Notice that \( \{B_2, B_3, \ldots, B_k\} \) is a set partition of \([n]\) which shows a bijection with the set partitions of \([n]\) whose blocks except for the \( B_1 \) do not contain two consecutive numbers and the set partitions of subsets \([n]\) \setminus \{1\} with no two consecutive numbers in the same block. By [13], \( |R_n([12])| \) is counted by the sequence \( A261041 \) in Sloane and this completes the proof.

According to Theorem 1, we are successful in enumerating RGFs avoiding an arbitrary single vincular pattern of length 2 and 3. The enumerations are relevant to classic sequences in Sloane [13]. As for vincular pattern of length 4, unfortunately by our computer programs, it does not show many results relevant to classic sequences. We have a few of them which are set as a conjecture below.

**Conjecture 1.** Let \( n \) be a positive number. Then \( |R_n([1212])| \) and \( |R_n([222])| \) are counted by the sequence \( A098569 \) in Sloane [13].

Next we enumerate RGFs avoiding a 2-pattern set of a vincular pattern and a classic pattern.

While the proofs in Theorem 1 are quite combinatorial using the characterizations for given RGF classes, the next ones are using techniques of generating functions integrated with succession rules and recursive trees. We recall the generating function for a sequence. Given a sequence, say \( (a_0, a_1, \ldots, a_n, \ldots) \). We can consider \( a_n \) as the number of objects of size \( n \) which we need to count. The generating function \( f(x) \) for \( (a_0, a_1, \ldots, a_n, \ldots) \) is the formal power series

\[
f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots,
\]

where \( x \) is a variable. Ignoring the convergence of the series, \( f(x) \) can be considered as a function in variable \( x \). Moreover,
in some cases we can find a reduced form for \( f(x) \). Notice that if generating functions for two sequences are the same then the two sequences are identical.

**Theorem 2.** Let \( n \) be a positive integer. Then, \( |R_n(122, 112)| \) is counted by Fibonacci sequence.

**Proof.** We first show that \( R_n(122, 112) \) is the set of RGFs starting with an initial run, following a strictly decreasing sequence and ending with occurrences of 0. Explicitly,

\[
R_n(122, 112) = \{01 \ldots am_1m_2 \ldots m_0 \cdot 0 \cdot 0 \in R_n \text{ such that } a > m_1 > m_2 > \cdots > m_k > 0\}.
\]

Thus, it is straightforward that the right-hand set is a subset of \( R_n(122, 112) \) since each its element avoids both 122 and 112. Now let \( s \) be a concatenation or initial run 012...a and a non-increasing sequence \( a \geq m_1 \geq m_2 \geq \cdots \geq 0 \). On the other hand, as \( s \) avoids 122, we have \( m_i > m_{i+1} \) for each \( m_{i+1} \neq 0 \), or otherwise \( s \) will contain \( 0m_1m_{i+1} \), which is of pattern 122.

Now, each element of \( R_n(122, 112) \) is a concatenation of an initial run 01...a, a strictly decreasing sequence whose components do not exceed \( a-1 \), and a sequence of 0. By some simple calculations, it is proved that the generating function for the sequence \( R_n(122, 112) \) is given by

\[
f(x) = \sum_{a=1}^{\infty} x^{a+1}(1+x)^{a-1} \frac{1}{1-x} + \sum_{i=1}^{\infty} x^i + 1,
\]

where the second sum represents the case \( a = 0 \), and the third sum represents the case \( n = 0 \). Reducing this formula, it gives

\[
f(x) = \frac{1}{1-x-x^2},
\]

which coincides to the generating function for Fibonacci numbers.

Next, we enumerate RGFs avoiding a pattern set of a vincular pattern of length 4 and 213 using the ECO method which has been developed in [1]–[3] by showing the succession rules for constructing corresponding recursive trees and generating functions.

**Theorem 3.** Let \( n \) be a positive integer.

1) Let \( f(x) \) be the generating function for the sequence

\[
\{a_n = |R_n(1212, 213)|, n = 0, 1, 2, \ldots \}.
\]

Then,

\[
f(x) = \frac{1-x}{1-3x-x^2}.
\]

Consequently, \( a_n \) is counted by the sequence A122367 in Sloane.

2) Let \( g(x) \) be the generating function for the sequence

\[
\{b_n = |R_n(1221, 213)| : n = 0, 1, 2 \ldots \}.
\]

Then,

\[
g(x) = \frac{3x^2 - 3x + 1}{(1-x)(1-2x)^2}.
\]

Consequently, \( b_n \) is counted by the sequence A005183 in Sloane.

In order to prove the theorem, we use ECO method developed in [1]–[3] by showing the succession rules for the sets of these RGFs.

Recall that the succession rule allows us to count the number of successors of children nodes upon on the number of successors of its father. More precisely,

- Succession rule is a sequence of rules assigning the label of a node to labels of its successors. Each node is labeled by a pair of integers \((k, c)\), denoted by \(k_c\), where \(k\) is equal to the number of successors of the node and \(c\) is its color which represents the different rules for nodes with the same number of successors \(k\).
- The succession rule shows its root which is the label of the object with smallest size.
- Each node with the same label \(k_c\) will generate \(k\) nodes of the same labels such as \(\{c_1^1, c_2^2, \ldots, c_k^k\}\). This does not depend on type of nodes but only their labels. More explicitly, a succession rule for a node labeled \(k_c\) can be written by

\[
(k_c) \rightsquigarrow (c_1^1)(c_2^2) \cdots (c_k^k),
\]

where \((c_i^j, c_i)\) is the label for the successor \(i\) of that node.

Notice that in some enumeration problems on regular combinatorial objects [3], node label is often chosen to be the number of its successors. Therein, there is a unique color for all nodes of the same number of successors. Next we use the method developed to objects with less regularity using labels as pairs of integers [2]. Accordingly, we construct succession rules for our interested RGF classes avoiding a two-pattern set including a vincular pattern and the classic pattern 213. These classes are counted by classic sequences in Sloane.

**Proposition 1.** a) The succession rule for \( R_n(1212, 213) \) is given by

\[
\begin{aligned}
&\{(2_0)\}, \\
&(\{2_0\} \rightsquigarrow (2_0)(3_0)), \\
&\Omega_1 = \{(1_0) \rightsquigarrow (1_0), \\
&(k_0) \rightsquigarrow (1_0)(2_1) \ldots ((k-2)1)((k+1)0), \quad \forall k \geq 3, \\
&(k_1) \rightsquigarrow (1_0)(2_1)(3_1) \ldots (k_1), \quad \forall k \geq 2.
\end{aligned}
\]

\(\) (1)

Here, \(2_0\) is the root label of the succession rule \(\Omega_1\).

b) The succession rule for \( R_n(1221, 213) \) is given by

\[
\begin{aligned}
&\{(2_1)\}, \\
&(\{2_1\} \rightsquigarrow (2_1)(c+2)_0), \quad \forall c \geq 1, \\
&\Omega_2 = \{(2_0) \rightsquigarrow (2_0)(1_0), \\
&(1_0) \rightsquigarrow (1_0), \\
&(k_0) \rightsquigarrow (2_0)(k-2)((2_0+1))(k+1)0), \quad \forall k \geq 3.
\end{aligned}
\]

\(\) (2)

Here, \(2_1\) is the root label of the succession rule \(\Omega_2\).

**Proof.** 1) Proving the succession rule is a routine by checking the number of successors for each successor of a
given RGF \( w \). We first label words \( w \in R_n(1212,213) \) to classify them as follows.

a) If \( w = 00 \ldots 0 \), then \( w \) is labeled by \( 2_0 \).

b) If \( w \neq 00 \ldots 0 \) and \( w \) ends with 0, then \( w \) is labeled by \((1)_0\).

c) If \( w 
eq 00 \ldots 0 \) and if \( w \) ends with a descent where the last letter is equal to \( k \), then \( w \) is labeled by \((k + 1)_1\).

d) If \( w \) ends with a plateau \( kk \) where \( k < \max w \), then \( w \) is labeled by \((k + 1)_1\).

e) If \( w \) ends with a plateau \( kk \), where \( k = \max w \), then \( w \) is labeled by \((k + 2)_0\).

f) If \( w \) ends with an ascent and the last letter \( k \), then \( w \) is labeled by \((k + 2)_0\). It is straightforward that \( w \) has \( k + 2 \) successors \( w_0,w_1,\ldots,w(k + 1) \).

We now show the succession rule presented in the statement by considering the number of successors for each successor of \( w \) upon the labeling of \( w \):

(i) \( w = 00 \ldots 0 \): It is trivial that \( w \) is labeled by \( 2_0 \) and that \( w \) has 2 successors \( 00 \ldots 0 \) of label \( 2_0 \) and \( 00 \ldots 01 \) of labeled \( 3_0 \) as indicated in cases (1c) and (1d) above.

(ii) \( w = w'0 \), where \( w' 
eq 00 \ldots 0 \). Notice that \( w \) has only one successor \( w_0 \). Thus, \( w_1 = w'01 \) contains the subsequence 0101 of pattern 1212, and \( w_k = w'0k \), for \( k \geq 2 \), contains a subsequence 10k of pattern 213.

(iii) \( w \) is labeled by \( k_1 \) (\( k \geq 2 \)). It is a routine to check that \( w \) has \( k \) successors \( w_0 \) of labeled \( 1_0 \), \( w_1 \) of label \( 2_1 \), etc., \( w(k - 1) \) of label \( k \) as indicated in cases (1c) and (1d) above.

(iv) \( w \) is labeled by \( k_0 \) (\( k \geq 2 \)). Then, \( w \) has \( k + 1 \) successors \( w_0, w_1, \ldots, w_k \) where

- \( w_0 \) of labeled \( 1_0 \);
- \( w_1, w_2, \ldots, w(k - 2) \) of labels \( 2_1, 3_1, \ldots, (k - 2)_1 \) respectively by the case (1c) above;
- \( w(k - 1), w_k \) of labels \( k_0 \) and \( (k + 1)_0 \) respectively by the case (1e) and (1f) above.

This completes the proof of the statement 1.

2) Similarly to the proof of the part 1, we will show the second statement by showing how we label words in \( R_n(1221,213) \) as follows

a) If \( w \) ends with a descent, then \( w \) is labeled by \( 2_0 \).

b) If \( w \) ends with a plateau and the last letter \( c \), then \( w \) is labeled by \( 2_{c+2} \).

c) If \( w \) ends with an ascent and the last letter \( k \), where \( k = \max w \) is the unique maximum in \( w \), then \( w \) is labeled by \( (k + 2)_0 \).

d) If \( w \) ends with an ascent and the last letter \( k \), where \( k \) is not the unique maximum element in \( w \), then \( w \) is labeled by \( (1)_a \).

The rest of the proof is a routine to check that these labels generate succession rules as indicated in the statement.

\[ \text{Proof of Theorem 3.} \] The theorem is followed by calculating the corresponding generating functions using the given succession rules and the recursive trees. By definition of the succession rule, all subtrees of the generating tree rooted at the same label are isomorphic. Figures 1 and 2 show the labeled trees corresponding the succession rule \( \Omega_1 \) and \( \Omega_2 \) respectively.

Now let \( T_{k_e}(x) \) be the generating function for the tree rooted at \( k_e \), i.e.,

\[ T_{k_e}(x) = \sum_{i=0}^{\infty} a_i x^i, \]

where \( a_i \) is the number of nodes at level \( i \) of the subtree rooted at \( k_e \). Notice that the generating function we need to find is equal to the generating function of the subtree with the root labeled by the root of the succession rule.

We have the relation between each rule of the succession rule and generating functions for corresponding subtrees as follows. Suppose that the succession rule for \( (k_e) \) is given by

\[ (k_e) \sim (e_{c_1})(e_{c_2}) \ldots (e_{c_{k_e}}). \]

Then, it is straightforward that

\[ T_{k_e}(x) = 1 + x(T_{c_1}(x) + T_{c_2}(x) + \cdots + T_{c_{k_e}}(x)). \]

Solving the generating functions corresponding to the succession rules given, we obtain

1) for the succession rule given by \( \Omega_1 \) in (1)

i) \( T_1(x) = \frac{1}{1-x} \)

ii) \( T_{k_1}(x) = \frac{1}{1-x} \)

iii) \( T_{k_0}(x) = T_{k_1}(x) + \frac{1}{1-x} T_{(k+1)_0}(x) \)

iv) \( f(x) = T_{k_0}(x) = \frac{1}{1-x} \).

2) for the succession rule given by \( \Omega_2 \) in (2)

i) \( T_1(x) = \frac{1}{1-x} \)

ii) \( T_{2_0}(x) = \frac{1}{(1-x)^2} \)

iii) \( T_{k_0}(x) = \frac{1}{(1-x)^2} + \frac{x(k-2)}{1-x} T_{(k+1)_0}(x) \)

iv) \[ T_{2_1}(x) = \frac{1}{1-x} \left(1 + \frac{x}{1-x} + 2 \left(1 + \frac{x}{1-x}\right)^2 + \cdots + n \left(1 + \frac{x}{1-x}\right)^n \right) \]

\[ + \frac{x}{1-x} \left(1 + \frac{x}{1-x}\right)^n \left(1 - \frac{x}{1-x}\right) \]

\[ \left(1 + \frac{x}{1-x}\right)^n T_{(n+2)_0}(x). \]

v) \( g(x) = T_{2_1}(x) = \frac{x_2+x_1+1}{(1-x)(1-x^2)^3} \).

These complete the proof.

\[ \text{III. Graycoodness on RGFs avoiding a Pattern Set} \]

Given two words \( s \) and \( t \) of the same length. We recall that Hamming distance between \( s \) and \( t \), denoted by \( d(s,t) \), is the number of distinct letters between \( s \) and \( t \). A finite set \( S \) of words is said to be \( d \)-Graycode if there exists a linear order on \( S \) such that Hamming distances between two any consecutive words by that order are at most \( d \). Investigating \( d \)-Graycoodness allows us to generate classes of words efficiently.
and has received much attention in the literature. In [10], [11], Sabri and Vajnovszki considered the reflected Gray Code order and proved the 3-Graycodeness for the set of RGFs of length \( n \) (with no pattern avoidance). Inherited from those works, in this section we develop the 3-Graycodeness for classes of RGFs avoiding pattern sets considered in Section II.

For each word \( s \), we denote by \( \Sigma(s) \) the summation of all its letters, and by \( \max(s) \) the maximum value of letters in \( s \). For instance, if \( s = 012130 \), then \( \Sigma(s) = 7 \) and \( \max(s) = 3 \).

The reflected Gray Code order on \( \mathbb{N}_0^n \) is defined as follows.

**Definition 2.** Let \( s, t \in \mathbb{N}_0^n \) and \( s \neq t \). Let \( p \) be the maximal common prefix of \( s \) and \( t \), and \( k \) be the length of \( p \). We say that \( s \) precedes \( t \) by the reflected Gray Code order, denoted by \( s < t \), if either \( \Sigma(p) \) is even and \( s_{k+1} < t_{k+1} \), or \( \Sigma(p) \) is odd and \( s_{k+1} > t_{k+1} \).

Clearly the reflected Gray Code order applies to \( R_n \) since \( R_n \subset \mathbb{N}_0^n \). For instance, let \( s = 01211034 \) and \( t = 0121345 \) be two RGFs. Then \( s \) and \( t \) share the maximal common prefix \( p = 01211 \) of length 5, and \( s < t \) as \( \Sigma(p) = 5 \) which is odd and the sixth letter of \( s \) is 0 smaller than the sixth letter of \( t \).

The following theorem presents a sufficient condition for 3-Graycodeness of RGFs avoiding a pattern set.

**Theorem 4.** Given a pattern set \( T \). If for each \( w \in R_n(T) \) for \( n = 1, 2, \ldots \), both \( w(0) \) and \( w(\max(w) + 1) \) are words in \( R_{n+1}(T) \), then \( R_n(T) \) is 3-Graycode by the reflected Gray Code order.

**Proof.** Let \( s \) and \( t \) be consecutive words in \( R_n(T) \) by the reflected Gray Code order and \( s < t \). Let \( p = s_1 s_2 \ldots s_k \) be the maximal common prefix of \( s \) and \( t \). We show that \( d(s, t) \leq 3 \) by showing that \( s_i = t_i = 0 \) for any \( i = k + 4, \ldots, n \).

Since \( s \) and \( t \) are consecutive by the reflected order, \( s \) (resp. \( t \)) must be the last (resp. first) word generated from \( ps_{k+1} \) (resp. \( pt_{k+1} \)) by the \( \prec \) order. Since it is always possible to insert the maximum and the minimum letters, by definition and math calculations, it is shown that

1) \( s \) is of the following forms

- \( p s_{k+1} M_1 (M_1 + 1) 0 \ldots 0, \) if \( \Sigma(p) + s_{k+1} \) is even and \( M_1 \) is even.
- \( p s_{k+1} M_1 0 \ldots 0, \) if \( \Sigma(p) + s_{k+1} \) is even, and \( M_1 \) is odd.
- \( p s_{k+1} 0 \ldots 0, \) if \( \Sigma(p) + s_{k+1} \) is odd.

2) \( t \) is of the following forms

- \( pt_{k+1} M_2 (M_2 + 1) 0 \ldots 0, \) if \( \Sigma(p) + t_{k+1} \) is odd, and \( M_2 \) is even.
- \( pt_{k+1} M_2 0 \ldots 0, \) if \( \Sigma(p) + t_{k+1} \) is odd, and \( M_2 \) is odd.
- \( pt_{k+1} 0 \ldots 0, \) if \( \Sigma(p) + t_{k+1} \) is even.

Therein, \( M_1 = \max\{\max(p), s_{k+1}\} + 1 \) and \( M_2 = \max(\max(p), t_{k+1}) + 1 \). Therefore, \( s_i = t_i = 0 \) for any \( i = k + 4, \ldots, n \) which shows that \( d(s, t) \leq 3 \).

**Corollary 2.** Let \( A_n \) be one of the following sets: \( R_
(122), R_{n}(122), R_{n}(1212), R_{n}(12212), R_{n}(12122) \). Then, \( A_n \) is 3-Graycode by the \( \prec \) order.

**Proof.** By Theorem 4, it is sufficient for checking the following facts for each \( s \in A_{n-1} \): then

- \( s(0) \in A_{n} \), and
- \( s(\max(s) + 1) \in A_{n} \).

Notice that it is sometimes easy to check the satisfaction of generating conditions in Theorem 4 as presented in Corollary 2. However, Theorem 4 only showed a sufficient but not a necessary condition the for 3-Graycodeness by the reflected Gray code order. Indeed, it can be proved that \( R_{n}(122, 112) \) is 3-Graycode using similar arguments as the proof of Theorem 4 while considering the difference between two consecutive words as from the common prefix. Moreover, \( R_{n}(122, 112) \) does not satisfy the generating conditions in Theorem 4. Particularly, \( w = 0121 \in R_{n}(122, 112), \) but \( w(\max(w) + 1) = 01213 \not\in R_{n}(122, 112) \) as it contains the subsequence 113 of pattern 112.

It would be interesting if we can show necessary conditions for vincular pattern sets such that the set of RGFs avoiding them is 3-Graycode.

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**REFERENCES**


Fig. 1: First levels of the labeled tree for RGFs avoiding 1212 and 213.
Fig. 2: First levels of the labeled tree for RGFs avoiding 1221 and 213.